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On Dynamic Programming in Economic Models Governed by DDEs

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A family of optimal control problems for economic models, where state variables are driven by delay differential equations (DDEs) and subject to constraints, is treated by Bellman's dynamic programming in infinite dimensional spaces. An existence theorem is provided for the associated Hamilton-Jacobi-Bellman (HJB) equation: the value function of the control problem solves the HJB equation in a suitable sense (although such value function cannot be computed explicitly). An AK model with vintage capital and an advertising model with delay effect are taken as examples.

Keywords: delay differential equations; dynamic programming

1. INTRODUCTION

We develop dynamic programming for a family of optimal control problems related to economic models described by DDEs. The presence of delays in the state equation complicates their treatment. To deal with this difficulty, we rewrite the DDEs as ODEs in a suitable Hilbert space, and use Bellman's dynamic programming in infinite dimensional spaces.

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The infinite dimensional problem has features that were not met together so far in a general formulation, namely, the presence in the state equation of non analytical unbounded operators, without further smoothing assumptions; the appearance of the delay in the state and in the control, rendering the control operator possibly unbounded; the constraints on the state and on the control, proper of economic problems. These difficulties are commonly met in economic models governed by DDEs.

The considered DDEs are linear and the objective functional is *concave*. In the absence of concavity, one can still apply dynamic programming in the framework of viscosity solutions, which we avoid here ("User's guide" by Crandall et al. (1985) for a standard reference on viscosity solutions). Fabbri and Gozzi (2008) studied an easier case enabling the explicit solution of the associated Hamilton-Jacobi-Bellman (HJB) equation, while here we develop dynamic programming when explicit solutions of the associated HJB equation are not available, and in finite time horizon. The main result is that the value function is a solution in a suitable weak sense of the HJB equation. This is a first step toward the so-called "Verification Theorem," which is powerful to study optimal paths.

The general model is motivated by two economic examples, an AK model with vintage capital (Boucekkine et al., 2004, 2005; Fabbri and Gozzi, 2008) and an advertising model with delay effects (Gozzi and Marinelli, 2004; Gozzi et al., 2006). In section 2.1 we present the examples; in section 3 we recall dynamic programming and overview the current literature on dynamic programming in infinite dimensional optimal control problems. In section 4 we rewrite the state equations as an ODE in a suitable Hilbert space. In section 5 we write the resulting infinite dimensional optimal control problem and its HJB equation. In section 6 we show our main result, the existence of an ultra-weak solution of the HJB equation.

2. TWO EXAMPLES

2.1. An AK Model with Vintage Capital

We consider the AK growth model with stratification of capital of Boucekkine et al. (2005), in finite time horizon and with a more general concave target functional. The model proves interesting in the study of short run fluctuations and of transitional dynamics: the reader is referred to Boucekkine et al. (2005) for the analysis, in the infinite horizon case.

The AK-growth model with vintage capital is based on the accumulation law for capital goods:

$$k(s) = \int_{s-R}^{s} i(\sigma)d\sigma \tag{1}$$

where $i(\sigma)$ is the investment at time σ. Capital goods are accumulated during R (scrapping time) and then dismissed. Investments are differentiated with age. If the production function is linear:

$$y(s) = ak(s) \tag{2}$$

where $y(s)$ is the output at time s (AK reminds us of the linear dependence of the dynamic from the trajectory – a constant A multiplied by K; the constant A is a here), and at every time the social planner splits the production into consumption $c(s)$ and investment $i(s)$:

$$y(s) = c(s) + i(s), \tag{3}$$

then Eq. (1) is rewritten into infinitesimal terms as:

$$\dot{k}(s) = ak(s) - ak(s-R) - c(s) + c(s-R), \quad s \in [t, T] \tag{4}$$

which is a DDE. The time variable s varies in $[t, T]$, with t the initial time and T the (finite) time horizon. The social planner maximizes the functional

$$\int_{t}^{T} e^{-ps} h_0(c(s))ds + \phi_0(k(T)) \tag{5}$$

where h_0 and ϕ_0 are concave upper-semi-continuous (u.s.c.) utility functions. In Boucekkine et al. (2005) the horizon is infinite and $\phi_0 = 0$. Here, the instantaneous utility is of constant relative risk aversion (CRRA); that is, the function h_0 is $h_0(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, where $\sigma \in [0, 1)$. The case when $\sigma \to 1$, that is $h_0(c) = \ln c$, is not considered, as the domain is not a closed convex subset of $\mathbb{R}$, as explained in section 5.

The capital at time s and consequently the production, and the consumption at time s cannot be negative:

$$k(s) \geq 0, \quad c(s) \geq 0, \quad \forall s \in [t, T] \tag{6}$$

These constraints are different from the more restrictive and more natural ones used by Boucekkine et al. (2005), where also the investment path $i(\cdot)$, given through Eq. (2), Eq. (3) by $i(\cdot) = ak(\cdot) - c(\cdot)$, was assumed positive. The main reason for our choice is technical: we do not have the existence of weak solutions (as later defined), for instance, with mixed constraints such as in Boucekkine et al. (2005).

In order to take the constraints into account, we assume that consumption, which is the control variable of the system, lies in the admissible set:

$$\mathcal{A} \stackrel{\text{def}}{=} \{c(\cdot) \in L^2([t,T],\mathbb{R})\colon\ c(\cdot) \geq 0 \text{ and } k(\cdot) \geq 0 \text{ a.e. in } [t,T]\}. \tag{7}$$

2.2. An Advertising Model with Delay Effects

Another example of optimal control problems driven by DDEs is the dynamic advertising model presented in the stochastic case by Gozzi et al. (2006) and by Gozzi and Marinelli (2004), and in the deterministic case by Faggian and Gozzi (2004) and Feichtinger et al. (1994) (with references therein).

For $t \geq 0$ an initial time, and $T > t$ a terminal time ($T < +\infty$), let $\gamma(s)$, with $0 < t \leq s \leq T$, represents the stock of advertising goodwill of the product to be launched. The dynamics is given by the DDE with delay $R > 0$ where z is the intensity of spending in advertisement:

$$\begin{cases} \dot{\gamma}(s) = a_0\gamma(s) + \int_{-R}^{0}\gamma(s+\xi)da_1(\xi) + b_0 z(s) + \int_{-R}^{0} z(s+\xi)db_1(\xi) \quad s \in [t,T] \\ \gamma(t) = x; \quad \gamma(\xi) = \theta(\xi), \quad z(\xi) = \delta(\xi), \quad \forall \xi \in [t-R,t], \end{cases} \tag{8}$$

under the assumptions:

- a_0 is a constant factor of image deterioration in the absence of advertising, $a_0 \leq 0$;
- $a_0(\cdot)$ is the distribution of forgetting time, $a_1(\cdot) \in L^2([-R,0];\mathbb{R})$;
- b_0 is a constant advertising effectiveness factor, $b_0 \geq 0$;
- $b_1(\cdot)$ is the density function of the time lag between the advertising expenditure z and the corresponding effect on the goodwill level, $b_1(\cdot) \in L^2([-R,0];\mathbb{R}_+)$;
- x is the level of goodwill at the beginning of the advertising campaign, $x \geq 0$;
- $\theta(\cdot)$ and $\delta(\cdot)$ are respectively the goodwill and the spending rate before the beginning, $\theta(\cdot) \geq 0$, with $(\theta(0) = x$, and $\delta(\cdot) \geq 0$.

When $a_1(\cdot)$, $a_1(\cdot)$ are zero, Eq. (8) reduces to the classical model contained in Nerlove and Arrow (1962). Goodwill and investment in advertising at each time s cannot be negative:

$$\gamma(s) \geq 0, \quad z(s) \geq 0, \quad \forall s \in [t,T]. \tag{9}$$

The objective functional is

$$J(t,x;z(\cdot)) = \varphi_0(\gamma(T)) - \int_t^T h_0(z(s))ds, \tag{10}$$

where φ_0 is a concave utility function, h_0 is a convex cost function, and the dynamic of γ is determined by Eq. (8). The functional J has to be maximized over admissible controls $\mathcal{U}$, for instance $L^2([t,T];\mathbb{R}_+)$, the space of square integrable nonnegative functions.

3. DYNAMIC PROGRAMMING

Dynamic programming (DP) to optimal control takes four steps (Fleming and Rishel (1975) in the finite dimensional case, and Li and Yong (1995) for the infinite dimensional case):

(i) let the initial data vary, we call *value function* the supremum of the objective functional and write an equation whose candidate solution is the value function: the so-called DP principle, together with its infinitesimal version, the Hamilton-Jacobi-Bellman (HJB) equation;
(ii) solve (whenever possible) the HJB equation to write the value function explicitly;
(iii) prove that the present value of the optimal control strategy can be expressed as a function of the present value of the optimal state trajectory: the closed loop, or feedback, optimal controls;
(iv) solve, if possible, the closed loop equation (CLE), which is the state equation where the control is replaced by its feedback expression obtained at step (iii): the solution is the optimal state trajectory and the optimal control strategy results from feedback law accordingly.

This method, when applicable, allows a powerful description of the optimal paths of an optimal control problem.

We clarify that the two models are not easy to manage with the DP approach because of two difficulties.

- The state equation is a DDE; DP is for controlled ODE. One way (Kolmanovskĭi and Shăikhet, 1996, for another way) is to rewrite the DDE as an ODE in an infinite dimensional space, which plays the role of the state space. We use the techniques developed by Delfour, Vinter and Kwong (section 4 and subsection 3.1 for references). The resulting infinite dimensional control problem is harder than the ones usually treated in Li and Yong (1995) because of the unboundedness of the control operator and the non-analyticity of the involved semigroup (subsection 4).

- There are state constraints, see Eq. (6) and Eq. (9). Such constraints complicate the problem already in finite dimension; in infinite dimension the general theory is not well established: only few results in special cases (different from the ones treated here) are known.

In the example of Section 2.1, when the time horizon is infinite and the utility function is a power, Fabbri and Gozzi (2008) overcame the difficulties and showed that an explicit solution of the HJB equation can be computed: the explicit formula allows the completion of dynamic programming.

Here, we do not write the utility function in a fixed explicit form (like the CRRA used by Boucekkine et al., 2005, and Fabbri and Gozzi, 2008), and so we do not obtain an explicit solution of HJB equation. We prove an existence theorem for the HJB equation, letting uniqueness, theoretical results of type (iii) and (iv), and numerical approximations as open problems.

3.1. References on Delay Differential Equations and on Dynamic Programming in Infinite Dimension

For delay differential equations a recent, interesting, and accurate reference is the book by Diekmann et al. (1995). The idea of writing delay system using a Hilbert space setting was first due to Delfour and Mitter (1972, 1975). Variants and improvements were proposed by Delfour (1986, 1980, 1984), Vinter and Kwong (1981), Delfour and Manitius (1977), Ichikawa (1977) and Bensoussan et al. (1992).

The optimal control problem in the linear-quadratic case is studied by Vinter and Kwong (1981), Ichikawa (1982), and Delfour et al. (1975). In the linear-quadratic case the Hamilton-Jacobi-Bellman reduces to the Riccati equation.

The study of the Hamilton-Jacobi-Bellman equation in Hilbert spaces began with Barbu and Da Prato (1983, 1985) and Barbu et al. (1983). A classical solution of HJB equations is differentiable in time and state, providing a more easy-to-handle closed loop form of the optimal strategy. Classical solutions are not always available, so that the point becomes the existence of "weak" solutions, or solutions that are possibly not differentiable.[1] We investigate the existence of a weak-type solution, which we call *ultra-weak*, in section 6 and which is limit of classical solutions of approximating equations. So far, to our knowledge, the existence of such solutions for the HJB equation associated to delayed differential state equations has not been studied, apart from the linear-quadratic case. In economics, the study of infinite dimensional optimal control problems dealing with vintage or heterogeneous capital or models of advertising is recent Barucci and Gozzi (1999), Feichtinger et al. (2006), Faggian (2005), Gozzi and Marinelli (2004), and Gozzi et al. (2006).

4. STATE EQUATION IN AN INFINITE DIMENSIONAL SETTING

We rewrite the state equations of our examples as controlled ODEs in a suitable Hilbert space.

4.1. Notation and Preliminary Results

We recall results on DDEs and on the related infinite dimension approach, referring the reader to Bensoussan et al. (1992) for more. From now on, $R>0$ and $a>0$ are fixed. Given $T>t\geq 0$ and $z\in L^2([t-R,T],\mathbb{R})$ (or $z\in L^2_{\text{loc}}([t-R,+\infty),\mathbb{R})$), for every $s\in[t,\ T]$ (or $s\in[t,+\infty)$) we call $z_s\in L^2([-R,0];\mathbb{R})$ the function

$$\begin{cases} z_s : [-R,0]\to\mathbb{R} \\ z_s(\sigma)\overset{\text{def}}{=}z(s+\sigma) \end{cases} \tag{11}$$

Given a control $c\in\mathcal{A}$ we consider the DDE:

$$\begin{cases} \dot{k}(s)=ak(s)-ak(s-R)-c(s)+c(s-R) \quad \text{for } s\in[t,T] \\ (k(t),k_t,c_t)=(\phi^0,\phi^1,\omega)\in\mathbb{R}\times L^2([-R,0];\mathbb{R})\times L^2([-R,0];\mathbb{R}) \end{cases} \tag{12}$$

where k_t and c_t are defined by means of Eq. (11). In the delay setting, the initial data are a triple made of the state, the history of the state, and the history of the control on the interval $[t-R,t]$. Eq. (12) does not

[1]The most general concept of weak solution is that of viscosity solution, introduced in the infinite dimensional case by Crandall and Lions Crandall et al. (1985).

make sense point-wise, and must be interpreted as we do in the sequel. We give now an existence result and estimate the solution:

Theorem 4.1. *Given an initial condition* $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R,0];\mathbb{R}) \times L^2([-R,0];\mathbb{R})$ *and a control* $c \in L^2([t,T],\mathbb{R})$ *there exists a unique solution k(·) of (12) in* $W^{1,2}([t,T],\mathbb{R})$. *Moreover there exists a positive constant* $C(T-t)$ *such that*

$$|k|_{W^{1,2}([t,T],\mathbb{R})} \le C(T-t)\Big(|\phi^0| + |\phi^1|_{L^2([-R,0];\mathbb{R})} + |\omega|_{L^2([-R,0];\mathbb{R})} + |c|_{L^2([t,T],\mathbb{R})}\Big) \tag{13}$$

Proof. Theorem 3.3, page 217 in Bensoussan et al. (1992) applies for the first part and Theorem 3.3 page 217, Theorem 4.1 page 222 and page 255 for the second statement.

In view of the continuous embedding $W^{1,2}([t,T],\mathbb{R}) \hookrightarrow C^0([t,T],\mathbb{R})$ we have also:

Corollary 4.2. *There exists a positive constant* $C(T-t)$ *(possibly different from the one defined in Theorem (4.1)) such that*

$$|k|_{C^0([t,T],\mathbb{R})} \le C(T-t)\Big(|\phi^0| + |\phi^1|_{L^2([-R,0];\mathbb{R})} + |\omega|_{L^2([-R,0];\mathbb{R})} + |c|_{L^2([t,T],\mathbb{R})}\Big) \tag{14}$$

We consider the continuous linear application L with norm $\|L\|$

$$\begin{aligned} &L\colon C([-R,0],\mathbb{R}) \to \mathbb{R} \\ &L\colon \varphi \mapsto \varphi(0) - \varphi(-R) \end{aligned} \tag{15}$$

and then define $\mathcal{L}^t$ as:

$$\begin{aligned} &\mathcal{L}^t\colon C_c([t-R,T],\mathbb{R}) \to L^2([t,T],\mathbb{R}) \\ &\text{where } \mathcal{L}^t(\phi)\colon s \mapsto L(\phi_s) \quad \text{for } s \in [t,T] \end{aligned} \tag{16}$$

where $C_c(t-R,T;\mathbb{R})$ is the set of real continuous functions having compact support contained in $(t-R,T)$.

Theorem 4.3. *The linear operator* $\mathcal{L}^t\colon C_c([t-R,T],\mathbb{R}) \to L^2([t,T],\mathbb{R})$ *has a continuous extension* $\mathcal{L}^t\colon L^2([t-R,T],\mathbb{R}) \to L^2([t,T],\mathbb{R})$ *with norm* $\le \|L\|$.

Proof. The proof is in Bensoussan et al. (1992) Theorem 3.3, page 217.

Using the "L" notation we rewrite Eq. (12) as

$$\begin{cases} \dot{k}(s) = aL(k_s) - L(c_s) \quad \text{for } s \in [t, T] \\ (k(t), k_t, c_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R, 0]; \mathbb{R}) \times L^2([-R, 0]; \mathbb{R}) \end{cases} \tag{17}$$

and using the "$\mathcal{L}^t$" notation we rewrite Eq. (12) as

$$\begin{cases} \dot{k}(s) = a(\mathcal{L}^t k)(s) - (\mathcal{L}^t c)(s) \quad \text{for } s \in [t, T] \\ (k(t), k_t, c_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R, 0]; \mathbb{R}) \times L^2([-R, 0]; \mathbb{R}) \end{cases} \tag{18}$$

So far, the history of the control and of the trajectory were kept distinct from one another.

The delay system depends jointly on those data, a fact which helps reduce the dimension of the state space.

- Given $u \in L^2([t-R, T], \mathbb{R})$ we define the function $e^t_+ u \in L^2([t-R, T], \mathbb{R})$ as:

$$e^t_+ u \colon [t-R, T] \to \mathbb{R}, \; e^t_+ u(s) = \begin{cases} u(s) & s \in [t, T] \\ 0 & s \in [t-R, t) \end{cases} \tag{19}$$

- Given $u \in L^2([-R, 0]; \mathbb{R})$ we define the function $e^0_- u \in L^2([t-R, T], \mathbb{R})$ as:

$$e^0_- u \colon [t-R, T] \to \mathbb{R}, \; e^0_- u(s) = \begin{cases} 0 & s \in [t, T] \\ u(s-t) & s \in [t-R, t) \end{cases} \tag{20}$$

- Given a function $u \in L^2([-R, 0]; \mathbb{R})$ and $s \in [t, T]$ we define the function $\eta(s)u \in L^2([-R, 0]; \mathbb{R})$ as:

$$\eta(s)u \colon [-R, 0] \to \mathbb{R}, \; (\eta(s)u)(\theta) = \begin{cases} u(-s+t+\theta) & \theta \geq -R+s-t \\ 0 & \theta < -R+s-t \end{cases} \tag{21}$$

As $k = e^t_+ k + e^0_- \phi^1$, and $c = e^t_+ c + e^0_- \omega$, we distinguish the solution $k(s),\ s \geq t$ and the control $c(s),\ s \geq t$ from the initial data ϕ^1 and ω:

$$\begin{cases} \dot{k} = a\mathcal{L}^t e^t_+ k - \mathcal{L}^t e^t_+ c + a\mathcal{L}^t e^0_- \phi^1 - \mathcal{L}^t e^0_- \omega \\ k(t) = \phi^0 \in \mathbb{R} \end{cases} \tag{22}$$

System (22) does not directly use the initial function ϕ^1 and ω but only the sum of their images $a\mathcal{L}^t e^0_+ \phi^1 - \mathcal{L}^t e^0_- \omega$. We introduce the operator

$$\begin{cases} \overline{L}: L^2([-R,0];\mathbb{R}) \to L^2([-R,0];\mathbb{R}) \\ (\overline{L}\phi^1)(\alpha) \stackrel{\text{def}}{=} L(\text{est}(\phi^1)_{-\alpha})) \quad \alpha \in (-R,0) \end{cases} \tag{23}$$

where est(ϕ^1) is the function $\mathbb{R} \to \mathbb{R}$ which goes to 0 out of $(-R,0)$ and which is equal to ϕ^1 in $(-R,0)$ (the same for ω).

The operator $\overline{L}$ is continuous (Bensoussan et al., 1992: 235) moreover

$$a\mathcal{L}^t e^0_- \phi^1(s) - \mathcal{L}^t e^0_- \omega(s) = (\eta(s)(a\overline{L}\phi^1 - \overline{L}\omega))(0) \quad \text{for } s \geq t. \tag{24}$$

If we set

$$x^1 \stackrel{\text{def}}{=} (a\overline{L}\phi^1 - \overline{L}\omega), \; x^0 \stackrel{\text{def}}{=} \phi^0, \tag{25}$$

then we rewrite Eq. (22) and consequently Eq. (12) as

$$\begin{cases} \dot{k}(s) = (a\mathcal{L}^t e^t_+ k)(s) - (\mathcal{L}^t e^t_+ c)(s) + (\eta(s)x^1)(0) \quad \text{for } s \geq t \\ k(t) = x^0 \in \mathbb{R} \end{cases} \tag{26}$$

where $\mathbb{R} \times L^2([-R,0];\mathbb{R}) \ni x \stackrel{\text{def}}{=} (x^0, x^1)$, $c \in \mathcal{A}$. Eq. (26) is meaningful for all $x \in \mathbb{R} \times L^2([-R,0];\mathbb{R})$, also when x^1 is not of the form of Eq. (25). We have embedded the original system (12) in a family of systems of the form of Eq. (26).

4.2. The State Equation of the AK Model in the Hilbert Setting

We work on the Hilbert space

$$M^2 \stackrel{\text{def}}{=} \mathbb{R} \times L^2([-R,0];\mathbb{R})$$

where the scalar product between two elements $\phi = (\phi^0, \phi^1)$ and $\xi = (\xi^0, \xi^1)$ is given by

$$\langle \phi, \xi \rangle_{M^2} \stackrel{\text{def}}{=} \langle \phi^1, \xi^1 \rangle_{L^2} + \phi^0 \xi^0. \tag{27}$$

We consider the homogeneous system

$$\begin{cases} \dot{z}(s) = (a\mathcal{L}^0 z)(s) \\ (z(0), z_0) = \phi \in M^2 \end{cases} \tag{28}$$

and define the family of continuous linear transformations on M^2

$$\begin{cases} S(s)\colon M^2 \to M^2 \\ \phi \mapsto S(s)\phi \stackrel{\text{def}}{=} (z(s), z_s). \end{cases} \tag{29}$$

Then $\{S(s)\}_{s\geq 0}$ is a C_0 semigroup on M^2 whose generator is

$$\begin{cases} D(G) = \{(\phi^0, \phi^1) \in M^2\colon \phi^1 \in W^{1,2}(-R, 0) \text{ and } \phi^0 = \phi^1(0)\} \\ G(\phi^0, \phi^1) = (aL\phi^1, D\phi^1) \end{cases} \tag{30}$$

where $D\phi^1$ is the first derivative of ϕ^1 (proof in Bensoussan et al. (1992), Chapter 4).

The second component ϕ^1 of the elements of $D(G)$ is in $C([-R, 0], \mathbb{R})$ so that with a slight abuse of notation, we re-define L on $D(G)$ as:

$$\begin{cases} L\colon D(G) \to \mathbb{R} \\ L(\phi^0, \phi^1) = L\phi^1 \end{cases} \tag{31}$$

Moreover, if $D(G)$ is endowed with the graph norm, we denote j the continuous inclusion $D(G) \hookrightarrow M^2$. The operators G and j are continuous from $D(G)$ into M^2 and L is continuous from $D(G)$ into $\mathbb{R}$. We call G^*, j^* and L^* their adjoints, and identify M^2 and $\mathbb{R}$ with their dual spaces, so that

$$\begin{aligned} &G^*\colon M^2 \to D(G)' \\ &j^*\colon M^2 \to D(G)' \\ &L^*\colon \mathbb{R} \to D(G)' \end{aligned} \tag{32}$$

are linear continuous.

Definition 4.4. The structural state $x(s)$ at time $t \geq 0$ is defined by

$$y(s) \stackrel{\text{def}}{=} (y^0(s), y^1(s)) \stackrel{\text{def}}{=} (k(s), a\overline{L}(e^t_+ k)_s - \overline{L}(e^t_+ c)_s + \eta(s)x^1) \tag{33}$$

We use y^0 and y^1 to indicate, respectively, the first and the second components of the structural state. A more explicit definition is: if we call $\overleftarrow{k}_s, \overleftarrow{c}_s \in L^2([-R, 0]; \mathbb{R})$ the applications

$$\begin{aligned} &\overleftarrow{k}_s\colon \theta \mapsto -k(s - R - \theta) \\ &\overleftarrow{c}_s\colon \theta \mapsto -c(s - R - \theta) \end{aligned} \tag{34}$$

the structural state can be written as

$$y(s) \stackrel{\text{def}}{=} (k(s), a\overleftarrow{k}_s - \overleftarrow{c}_s + \eta(s)x^1). \tag{35}$$

We write the delay equation in the Hilbert space M^2 by means of the theorem:

Theorem 4.5. *Let $y^0(s)$ be the solution of system (26) for $x \in M^2$, $c \in \mathcal{A}$ and let y(t) be the structural state defined in Eq. (33). Then for each $T \geq 0$, the state y is the unique solution in*

$$\left\{ f \in C([t,T], M^2) \colon \frac{\mathrm{d}}{\mathrm{d}s} j^* f \in L^2([t,T], D(G)') \right\} \tag{36}$$

to the equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} y(s) = G^* y(s) + L^* c(s) \\ y(t) = x. \end{cases} \tag{37}$$

Proof. Proof in Bensoussan et al. (1992) Theorem 5.1 Chapter 4.

4.3. State Equation of the Model of Advertising in the Hilbert Setting

Similar arguments are used for the model of advertising. We write here only the results. We call N and B the continuous linear functionals given by

$$\begin{aligned} &N \colon C([-R,0]) \to \mathbb{R} \\ &N \colon \varphi \mapsto a_0 \varphi(0) + \int_{-r}^{0} \varphi(\xi) d a_1(\xi) \end{aligned} \tag{38}$$

$$\begin{aligned} &B \colon C([-R,0]) \to \mathbb{R} \\ &B \colon \varphi \mapsto b_0 \varphi(0) + \int_{-r}^{0} \varphi(\xi) d b_1(\xi) \end{aligned} \tag{39}$$

Let G be the generator of C_0-semigroup defined as:

$$\begin{cases} D(G) = \{(\phi^0, \phi^1) \in M^2 \colon \phi^1 \in W^{1,2}(-R,0) \text{ and } \phi^0 = \phi^1(0)\} \\ G(\phi^0, \phi^1) = (N\phi^1, D\phi^1) \end{cases} \tag{40}$$

We define $\overline{N}$ and $\overline{B}$ in the same way we defined $\overline{L}$ in Eq. (23). We write the model of advertising in infinite dimensional form:

- The structural state in the model of advertising has the expression:

$$y(t) = (y^0(s), y^1(s)) \stackrel{\text{def}}{=} (\gamma(s), \overline{N}(e^0_+\gamma)_s - \overline{B}(e^0_+z)_s + \eta(s)x^1) \tag{41}$$

  where $x_1 = \overline{N}(\theta) - \overline{B}(\delta)$.
- The state equation becomes

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}y(s) = G^*y(s) + B^*z(s) \\ y(t) = x. \end{cases} \tag{42}$$

5. THE TARGET FUNCTIONAL AND THE HJB EQUATION

We rewrite the profit functional for the first example. A similar reformulation holds for the target functional of the second example. We consider a control system governed by the linear equation described in Theorem 4.5. The set of admissible controls is defined by

$$\mathcal{A} \stackrel{\text{def}}{=} \{c(\cdot) \in L^2([t,T], \mathbb{R}) \colon\ c(\cdot) \geq 0 \text{ and } y^0(\cdot) \geq 0\} \tag{43}$$

As usual, the trajectory $y(\cdot)$ (and then $y^0(\cdot)$) depends on the choice of the control $c(\cdot)$, and of initial time and state $y(\cdot) = y(\cdot; t, x, c(\cdot))$.

To apply the results in Faggian (2008b) and recalled in the Appendix, we reformulate the maximization problem as a minimization problem. We take the constraints into account by modifying the target functional. If h_0 and ϕ_0 are the concave u.s.c. functions appearing in Eq. (5), we define

$$\begin{aligned} &h\colon \mathbb{R} \to \overline{\mathbb{R}} \\ &h(c) = \begin{cases} -h_0(c) & \text{if } c \geq 0 \\ +\infty & \text{if } c < 0 \end{cases} \end{aligned} \tag{44}$$

$$\begin{aligned} &\phi\colon \mathbb{R} \to \overline{\mathbb{R}} \\ &\phi(r) = \begin{cases} -\phi_0(r) & \text{if } r \geq 0 \\ +\infty & \text{if } r < 0 \end{cases} \end{aligned} \tag{45}$$

Moreover, we set

$$\begin{aligned} &g\colon \mathbb{R} \to \overline{\mathbb{R}} \\ &g(r) = \begin{cases} -0 & \text{if } r \geq 0 \\ +\infty & \text{if } r < 0 \end{cases} \end{aligned} \tag{46}$$

Both h, ϕ and g are convex l.s.c. functions on $\mathbb{R}$. We define the *target functional* as

$$J(t,x,c(\cdot)) = \int_t^T e^{-\rho s}[h(c(s)) + g(y^0(s))]\mathrm{d}s + \phi(y^0(T)) \qquad (47)$$

with c varying in the *set of admissible controls* $L^2([t,T],\mathbb{R})$. Maximizing Eq. (5) in the class $\mathcal{A}$ is equivalent to minimizing J on the whole space $L^2([t,T],\mathbb{R})$. The original maximization problem for the AK-model was reformulated as the *abstract minimization problem*:

$$\inf\{J(t,x,c(\cdot)) : c \in L^2([t,T],\mathbb{R}),\ \text{and } y \text{ satisfies Eq. (38)}\}, \qquad (48)$$

Moreover, the HJB equation is associated to such minimization problem by DP, and given by

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x\rangle - F(t,\nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi_0(x) \end{cases} \qquad \text{(HJB)}$$

with F defined as:

$$\begin{cases} F\colon [0,T]\times D(G) \to \mathbb{R} \\ F(t,p) \stackrel{\text{def}}{=} \sup_{c\geq 0}\{-L(p)c - e^{-\rho t}h_0(c)\} = e^{-\rho t}h^*(-e^{\rho t}L(p)) \end{cases} \qquad (49)$$

where h^* is the Legendre transform of the convex function h. We refer to F as the Hamiltonian of the system.[2]

The abstract framework is set, and we address dynamic programming.

6. THE VALUE FUNCTION AS ULTRA-WEAK SOLUTION OF HJB

We define the value function of the optimal control problem described as

$$W(t,x) \stackrel{\text{def}}{=} \inf_{c(\cdot)\in L^2([t,T];\mathbb{R})} J(t,x,c(\cdot)). \qquad (50)$$

Our objective is to provide a suitable concept of solution of HJB, so that the value function V is a solution in this sense.

If the data satisfy certain assumptions involving convexity, semicontinuity, and coercivity of h, Faggian (2008b) showed that the value function of an optimal control problem with state constraints of type

[2]In the usual definition, the Hamiltonian should be $\langle p, G^*x\rangle - F(t,p) + e^{-\rho t}g(x)$.

Eq. (48) is the unique *weak* solution to a HJB equation of type Eq. (HJB), as recalled in the Appendix, Theorem 7.7. Coercivity for the function h is lacking in our case, as for the prototype of h_0 is a CRRA function with $\sigma \in [0, 1)$, which is sub-linear on the positive real axis. This causes the Hamiltonian of the problem, related to the Legendre transform of h_0, to be possibly nonregular, so that all previous definitions of solutions cannot be used. As stated in Appendix, a weak solution is limit of strong solutions of approximating equations, while a strong solution is itself limit of classical solutions of approximating equations. These concepts require the Hamiltonian to be differentiable with respect to the co-state variable p.

We are about to define an *ultra-weak* solution as the limit of weak solutions to Eq. (HJB). The concept of solution is generalized, although not in the same direction as before, due to possibly non regular Hamiltonians.

According to the notation used by Faggian (2005), if X and Y are Banach spaces, we set

$$\begin{aligned}
&\mathrm{Lip}(X;Y) = \left\{ f : X \to Y : [f]_L := \sup_{x,y \in X,\, x \neq y} \frac{|f(x) - f(y)|_Y}{|x - y|_X} < +\infty \right\} \\
&C^1_{\mathrm{Lip}}(X) := \{ f \in C^1(X) : [f']_L < +\infty \} \\
&C_p(X,Y) := \left\{ f : X \to \mathbb{R} : |f|_{C_p} := \sup_{x \in X} \frac{|f(x)|_Y}{1 + |x|^p_X} < +\infty \right\}, \\
&C_p(X) := C_p(X, \mathbb{R}).
\end{aligned}$$

Moreover we set

$$\Sigma_0(X) := \{ w \in C_2(X) \ : \ w \text{ is convex},\ w \in C^1_{\mathrm{Lip}}(X) \}$$

$$\begin{aligned}
\mathcal{Y}([0,T] \times X) = \{ & w : [0,T] \times X \to \mathbb{R} \ : \ w \in C([0,T], C_2(X)), \\
& w(t,\cdot) \in \Sigma_0(X),\ \nabla w \in C([0,T],\ C_1(X, X')) \}.
\end{aligned}$$

Definition 6.1. We say that a function V is an ultra-weak solution to

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x \rangle - F(t, \nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi_0(x) \end{cases} \tag{51}$$

if there exists a sequence $\{F_n\}_n$ of functions in the space $\mathcal{Y}([0,T] \times D(G))$ such that $F_n \uparrow F$ pointwise, and

$$V(t,x) = \lim_{n \to +\infty} V_n(t,x) = \inf_{n \geq 0} V_n(t,x) \tag{52}$$

with V_n the unique weak solution to

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x \rangle - F_n(t, \nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi_0(x) \end{cases} \tag{53}$$

Any weak solution V is convex in the state variable x, but not necessarily *l.s.c* in (t, x). The existence result for Eq. (HJB) comes from the proof that the value function of the control problem set in the previous section is an ultra-weak solution.

Theorem 6.2. *The value function W of the optimal control problem (48) is an ultra-weak solution of Eq. (HJB).*

Proof. We need to build a sequence of Hamiltonians F_n having the properties required by Definition 6.1. We choose

$$F_n(t,p) := e^{-\rho t} h_n^*(-e^{\rho t} L(p)) \tag{54}$$

with

$$h_n(c) = h(c) + \frac{1}{2n}|c|^2, \; n \in \mathbb{N}. \tag{55}$$

If we denote with $S_n f(x) = \inf_{y \in \mathbb{R}} \left\{ f(y) + \frac{n}{2}|x-y|^2 \right\}$ the Yosida approximation of a function f, then $[S_n f]^*(x) = f^*(x) + \frac{1}{2n}|x|^2$, so that

$$h_n^*(c) = S_n(h^*)(c). \tag{56}$$

The functions h_n^* being the Yosida approximations of a l.s.c. convex function, they are Frechét differentiable with Lipschitz gradient, with Lipschitz constant $[(h_n^*)']_L \le n$. Moreover, as h_n is a decreasing sequence, F_n is then increasing, as required by definition 6.1. Hence the assumptions in Theorem 7.7 are satisfied for the problem of minimizing the functional

$$J_n(t,x,c) = J(t,x,c) + \frac{1}{2n}\int_t^T e^{-\rho s}|c(s)|^2 ds \tag{57}$$

in $L^2([t,T],\mathbb{R})$ and the consequence is:

Lemma 6.3. *Let*

$$W_n(t,x) \stackrel{\text{def}}{=} \inf_{c \in L^2([t,T],\mathbb{R})} J_n(t,x,c), \tag{58}$$

be the value functions of the approximating optimal control problem. Then W_n is convex in x and l.s.c. in x and t, and it is the unique weak solution of

$$\begin{cases} \partial_t v(t,x) + \langle \nabla v(t,x), G^* x \rangle - F_n(t, \nabla v(t,x)) + e^{-\rho t} g(x) = 0 \\ v(T,x) = \phi(x) \end{cases} \tag{59}$$

Moreover there exists $c_n^* \in L^2([t,T],\mathbb{R})$ optimal for the approximating problems, that is, $W_n(t,x) = J_n(t,x,c_n^*)$.

To complete the proof we need to show that $W_n(t,x) \downarrow W(t,x)$.

Lemma 6.4. *The value function of Eq. (48) is given by*

$$W(t,x) = \lim_{n\to\infty} W_n(t,x) = \inf_n W_n(t,x). \tag{60}$$

Proof. By definition of J_n, for all t, x and n we have $J_n(t,x,c) \geq J_{n+1}(t,x,c)$ for all admissible controls c, so that

$$W_n(t,x) \geq W_{n+1}(t,x), \tag{61}$$

and $\{W_n(t,x)\}_n$ is a decreasing sequence. Subsequently, an ultra-weak solution V of HJB exists, and it is given by

$$V(t,x) \stackrel{\text{def}}{=} \lim_{n\to\infty} W_n(t,x) = \inf_{n\in\mathbb{N}} W_n(t,x). \tag{62}$$

We show that such a solution V necessarily coincides with W. As

$$J(t,x,c) \leq J_n(t,x,c), \quad \forall c \in L^2([t,T],\mathbb{R}), \tag{63}$$

by taking the infimum and then passing to limits, we obtain

$$W(t,x) \leq V(t,x). \tag{64}$$

We prove the reverse inequality. Let $\varepsilon > 0$ be arbitrarily fixed, and c_ε be an ε-optimal control for the problem, that is $W(t,x) + \varepsilon > J(t,x,c_\varepsilon)$. By passing to limits as $n \to +\infty$ in

$$V(t,x) \leq W_n(t,x) \leq J_n(t,x,c_\varepsilon) \tag{65}$$

one obtains

$$V(t,x) \leq J(t,x,c_\varepsilon) < W(t,x) + \varepsilon, \tag{66}$$

which together with Eq. (64) implies the statement.

We proved the lemma and Theorem 6.2.

We do not derive any uniqueness result for ultra-weak solutions. If for instance one tries to get uniqueness by showing that any ultra-weak solution of HJB is the value function of a certain control problem, difficulties arise, due to the fact that, although $h_n^* \uparrow \mathcal{H}$ if and only if there exists some h such that $h_n \downarrow h$, in general $\mathcal{H}^* \neq h$ unless some minimax condition is satisfied, such as

$$h = \inf_n \sup_r \{cr - h_n^*(r)\} = \sup_r \inf_n \{cr - h_n^*(r)\} = \mathcal{H}^*, \tag{67}$$

which is false in general.

7. APPENDIX

We recall the abstract framework and the main results obtained by Faggian (2005, 2008b), regarding strong and weak solutions of HJB.

In the abstract setting of Faggian (2005, 2008b) on some state space V', if H is a separable Hilbert space, A_0 is the generator of a strongly continuous semigroup of operators on H, and V is the Hilbert space $D(A_0^*)$ endowed with the scalar product $(v|w)_V := (v|w)_H + (A_0^* v| A_0^* w)_H$, then we set V' equal to its dual space endowed with the operator norm. The semigroup generated by A_0 is extended in a standard way to a semigroup $\{e^{As}\}_{s\geq 0}$ on the space V', with generator A, a proper extension of A_0.

We assume that the state equation in V' is given by

$$\begin{cases} y'(s) = Ay(s) + Bc(s), & s \in [t, T] \\ y(t) = x \in V' \end{cases} \tag{68}$$

with control operator $B \in L(U, V')$ (although $B \notin L(U, H)$, where U is the control space and $c \in L^2([t, T], U)$ the control. This equation can be expressed in mild form as

$$y(s) = e^{A(s-t)}x + \int_t^s e^{A(s-\sigma)} Bc(\sigma) d\sigma. \tag{69}$$

The role of V' in the case of the delay equation is played by the space $D(G)'$ and the role of A_0 by the operator G^*.

Besides, we consider a target functional J_0, associated to the state equation, of type

$$J(t, x, c) = \int_t^T [g(s, y(s)) + h(s, c(s))] ds + \varphi(y(\tau)) \tag{70}$$

with $h(t, \cdot)$ real, convex, l.s.c., coercive, and $g(t, \cdot)$ and ν real, convex, and $C^1(V')$ (respectively, l.s.c. in V' in the x variable, as more precisely stated in the next subsections. The problem is that of minimizing $J(t,x,\cdot)$ over the set of admissible controls $L^2([t,T];U)$.

In applications, the target functional is rather of type

$$J_0(t,x,c) = \int_t^T [\xi(s,y(s)) + \eta(s,c(s))]ds + \nu(y(T))$$

with $\eta(t,\cdot)$ real, convex, l.s.c., coercive, and $\xi(t,\cdot)$ and ν real, convex, and $C^1(H)$ (respectively, l.s.c. in H) in the x variable, defined on H, but not necessarily on V'. We need to assume that ξ and ν allow C^1 (respectively, l.s.c.) extensions $g(t,\cdot)$ and ϕ on the space V'. The existence of such extensions is of course a strong assumption, as commented by Faggian (2005).

Moreover, the value function is defined as

$$W(t,x) = \inf_{c\in L^2([t,T];U)} J(t,x,c), \tag{71}$$

Finally, we considered the (backward) HJB equation associated to the problem set in $[0,\ T] \times V'$

$$\begin{cases} v_t(t,x) - \mathcal{H}(t,B^*\nabla v(t,x)) + \langle Ax|\nabla v(t,x)\rangle + g(t,x) = 0, \\ v(T,x) = \varphi(x), \end{cases} \tag{72}$$

for all t in $[0,\ T]$ and x in $D(A)$ (indeed for all x in V'), where

$$\mathcal{H}(t,c) = [h(t,\cdot)]^*(-c). \tag{73}$$

$\mathcal{H}$ is well defined only for p in V, that is a proper subspace of H, to which $\nabla v(t,x)$ (the spatial gradient of v) belongs.

With such a problem in mind, we investigate existence and uniqueness for the forward HJB equation:

$$\begin{cases} \phi_t(t,x) + F(t,\nabla\phi(t,x)) - \langle Ax,\nabla\phi(t,x)\rangle = g(T-t,x), (t,x) \in [0,T]\times V' \\ \phi(0,x) = \varphi(x). \end{cases} \tag{74}$$

Such a HJB is the forward version of Eq. (72) if we set

$$F(t,p) := \mathcal{H}(t,B^*p) = \sup_{c\in U}\{(-Bc|p)_U - h(t,c)\}$$

7.1. Regular Data and Strong Solutions of HJB Equations

We first treat the case of regular data, from which the concept of strong solution originates.

Assumptions 7.1.

1. A: $D(A) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{sA}\}_{s\geq 0}$ on V';
2. $B \in L(U, V')$;
3. there exists $\omega > 0$ such that $|e^{\tau A}x|_{v'} \leq Me^{\omega\tau}|x|_{v'}, \forall \tau \geq 0$;
4. $F \in \mathcal{Y}([0,T] \times V), F(t,0) = 0, \sup_{t\in[0,T]}[F_p(t,\cdot)]_L < +\infty$;
5. $g \in \mathcal{Y}([0,T] \times V'), t \mapsto [g_x(t,\cdot)]_L \in L^1(0,T)$;
6. $\varphi \in \Sigma_0(V')$;
7. $h(t,\cdot)$ is convex, lower semi-continuous, $\partial_c h(t,\cdot)$ is injective for all $t \in [0,T]$;
8. $\mathcal{H} \in \mathcal{Y}([0,T] \times U), \mathcal{H}(t,0) = 0$ and $\sup_{t\in[0,T]}[\mathcal{H}_c(t,\cdot)] < +\infty$.

Definition 7.2. Under assumptions 7.1., we say that $\phi \in C([0,T], C_2(V'))$ is a strong solution of Eq. (74) if there exists a family $\{\phi^\varepsilon\}_\varepsilon \subset C([0,T], C_2(V'))$ such that:

(i) $\phi^\varepsilon(t,\cdot) \in C^1_{\text{Lip}}(V')$ and $\phi^\varepsilon(t,\cdot)$ is convex for all $t \in [0,T]; \phi^\varepsilon(0,x) = \varphi(x)$ for all $x \in V'$.
(ii) there exist constants $\Gamma_1, \Gamma_2 > 0$ such that

$$\sup_{t\in[0,T]} [\nabla\phi^\varepsilon(t)]_L \leq \Gamma_1, \quad \sup_{t\in[0,T]} |\nabla\phi^\varepsilon(t,0)|_V \leq \Gamma_2, \quad \forall \varepsilon > 0; \tag{75}$$

(iii) for all $x \in D(A)$, $t \mapsto \phi^\varepsilon(t,x)$ is continuously differentiable;
(iv) $\phi^\varepsilon \to \phi$, *as* $\varepsilon \to 0+$, *in* $C([0,T], C_2(V'))$;
(v) there exists $g_\varepsilon \in C([0,T]; C_2(V'))$ such that, for all $t \in [0,T]$ and $x \in D(A)$,

$$\phi^\varepsilon_t(t,x) - F(t, \nabla\phi^\varepsilon(t,x)) + \langle Ax, \nabla\phi^\varepsilon(t,x)\rangle_{V'} = g_\varepsilon(T-t,x) \tag{76}$$

with $g_\varepsilon(t,x) \to g_0(t,x)$, and $\int_0^T |g_\varepsilon(s) - g_0(s)|_{C_2} ds \to 0, as\ \varepsilon \to 0+.$

The main result contained in the work by Faggian (2005) is:

Theorem 7.3. *Under assumptions (7.1), there exists a unique strong solution* ϕ *of Eq. (74) in the class* $C([0,T], C_2V'))$ *with the properties:*

(i) for all $x \in D(A)$, $\phi(\cdot,x)$ is Lipschitz continuous;
(ii) $\phi(t,\cdot) \in \Sigma_0(V')$, for all $t \in [0,T]$.

Faggian (2008a) proved:

Theorem 7.4. *Under assumptions (7.1), with $F(t,p) := \mathcal{H}(t, B^*p)$, let W be the value function of the control problem, and let ϕ be the strong solution of Eq. (74) described in Theorem 7.3. Then*

$$W(t,x) = \phi\ (T-t,x), \quad \forall t \in [0,T],\ \forall x \in V', \tag{77}$$

that is, the value function W of the optimal control problem is the unique strong solution of the backward HJB Eq. (72).

7.2. Semi-continuous Data and Weak Solutions of HJB Equations

We treat the case of merely semi-continuous data, from which the concept of *weak* solution originates.

Assumptions 7.5. If K is a convex closed subset of V', we define

$$\Sigma_K \equiv \Sigma_K(V') := \{\phi : V' \to (-\infty, +\infty] : \phi \text{ is convex and } l.s.c., K \subset D(\phi)\}$$

where $D(\phi) = \{x \in V' : \phi(x) < +\infty\}$, and assume:

1. $C : D(C) \subset V' \to V'$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{sA}\}_{s\geq 0}$ *on* V';
2. $B \in L(U, V')$;
3. there exists $\omega > 0$ such that $|e^{sC}x|_{v'} \leq e^{\omega s}|x|_{v'}$, $\forall s \geq 0$;
4. $F \in \mathcal{Y}([0,T] \times V)$, $F(t,0) = 0$, $\sup_{t\in[0,T]}[F_p(t,\cdot)]_L < +\infty$;
5. $g(t,\cdot) \in \Sigma_K(V')$, for all $t \in [0,T]$; $g(\cdot,x)$ $l.s.c.$ and $L^1(0,T)$ for all $x \in V'$;
6. $\varphi \in \Sigma_K(V')$;
7. $h(t,\cdot)$ is convex, lower semi-continuous, $\partial_c h(t,\cdot)$ is injective for all $t \in [0,T]$; moreover $h(t,c) \geq a(t)|c|_U^2 + b(t)$, with $a(t) \geq A(T) > 0$, $b \in L^1(0,T;\mathbb{R})$;
8. $\mathcal{H} \in \mathcal{Y}([0,T] \times U)$, $\mathcal{H}(t,0) = 0$, *and* $\sup_{t\in[0,T]}[\mathcal{H}_c(t,\cdot)]_L < +\infty$.

Definition 7.6. Let $K \subset V'$ be a convex set, and let $\varphi \in \Sigma_K$ and $g(t,\cdot) \in \Sigma_K$ for all t in $[0,T]$. Then $\phi : [0,T] \times V' \to (-\infty, +\infty]$ is a weak solution of (HJB) if:

(i) $\phi(t,\cdot) \in \Sigma_K,\ \forall t \in [0,T]$;
(ii) there exist sequences $\{\varphi_n\}_n \subset \Sigma_0$ and $\{g_n\} \subset \mathcal{Y}([0,T] \times V')$, such that

$$\varphi_n(x) \uparrow \varphi(x), \quad g_n(t,x) \uparrow g(t,x), \quad \forall x \in V', \quad \forall t \in [0,T], \quad as\, n \to +\infty. \tag{78}$$

and moreover, if ϕ_n is the unique strong solution of

$$\begin{cases} \phi_t(t,x) + F(t, \nabla\phi(t,x)) - \langle Ax, \nabla\phi(t,x)\rangle\ v' = g_n(t,x)\ (t,x) \in [0,T] \times V' \\ \phi(0,x) = \varphi_n(x) \end{cases} \tag{79}$$

in $C([0,T], C_2(V'))$, then

$$\phi_n(t,x) \uparrow \phi(t,x), \quad \forall (t,x) \in [0,T] \times V'. \tag{80}$$

where $cl_{v'}(C)$ denotes the convex closure in the V'-norm of a subset C of H.

As strong solutions were proved by Faggian (2005) to be Lipschitz with respect to the time variable and C^1 with respect to the space variable, and the weak solution ϕ is a sup-envelop of strong solutions ϕ_n, then ϕ is lower semi-continuous in $[0,T] \times V'$. For the same reason ϕ_n convex in the x variable implies that ϕ is convex in x as well.

The role of the convex set K is played in the first example by the set

$$K \stackrel{\text{def}}{=} cl_{V'}(\{(x_0, x_1) : x_0 \geq 0\}) \tag{81}$$

Theorem 7.7. *Under Assumptions 7.5, with:*

$$g(t,x) := e^{-\rho t} g_0(x), \quad h(t,c) := e^{-\rho t} h_0(x). \tag{82}$$

the properties are equivalent:

(i) *there exists a unique weak solution of Eq. (74)*;
(ii) *at each $(t,x) \in [0,T] \times K$ there exists an admissible control.*

Moreover if (i) or (ii) holds, there exists an optimal pair (c^*, y^*) and

$$\phi(T-t,x) = J(t,x,c^*). \tag{83}$$

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